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A new duality approach to solving concave vector maximization problems

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Abstract We introduce a special class of monotonic functions with the help of support functions and polar sets, and use it to construct a scalarized problem and its dual for a vector optimization problem. The dual construction allows us to develop a new method for generating weak efficient solutions of a concave vector maximization problem and establish its convergence. Some numerical examples are given to illustrate the applicability of the method.

Keywords Duality · Polar set · Multiobjective problem · Weak efficient solution

AMS Subject Classification 90C31

1 Introduction

Let Y be a real normed space, Y' its topological dual and $\langle\cdot,\cdot\rangle$ the pairing between them. The space Y is partially ordered by a pointed and convex cone $C \subseteq Y$ with a nonempty interior. The nonnegative polar cone of C, denoted by C^+, is defined by $C^+ := \{\xi \in Y' : \langle \xi, y\rangle \geq 0 \quad \text{for all } y \in C\}$. Let X be a nonempty subset of a topological vector space, and let f be a mapping from X to Y. The vector (or multiobjective) optimization problem that we are going to study in this paper is the following weak maximization problem:

$$\begin{aligned} &\text{WMax} \quad f(x), \\ &\text{s.t.} \quad x \in X, \end{aligned} \tag{VP}$$

which means finding a point $x_0 \in X$ such that $f(x) \notin f(x_0) + \operatorname{int} C$ for every $x \in X$. Such a point x_0 is traditionally called a weak efficient solution and the set of all weak efficient solutions of (VP) is denoted by $S(\text{VP})$.

In this paper we are interested in finding the whole set $S(\text{VP})$ and its image $f(S(\text{VP}))$. This problem is important in multicriteria decision making, multicriteria design engineering

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and in many other applications (see [8, 18, 20]). To date there exist a huge number of methods to solve multiobjective problems; the interested reader can find a short description of various existing methods in the monograph [15]. However, as far as we know, apart from the linear case, there exist quite few works which offer methods for obtaining the whole solution set of (VP) and most of the theoretical results that are aimed at doing it are difficult to implement. In this paper, we wish to present a constructive method to generate the solution set of problem (VP) when X is a convex set and f is a concave function. It is known that the entire solution set of a concave vector maximization problem can theoretically be obtained by maximizing the scalar composite functions $\xi \circ f$ on X when ξ runs over a base Λ of the polar cone C^+. The choice of a finite subset of Λ to perform the computing is, however, so complex that even for linear problems one may never reach the solution set of the vector problem when that subset grows up to a dense subset of Λ (see [14] for an example of this situation and a discussion on some numerical methods of recent literature on finding the solution sets of linear and nonlinear multiobjective problems, see [2, 3, 5, 9, 16]). The main idea of our method is the following. First, we construct a sequence of scalarized problems and their duals, which have simple structure and are easier to deal with. Then, using the duality relation between the scalarized problems and their duals, we compute the solution sets of the scalarized problems and show that they are parts of and converge to the solution set of the vector problem (VP). We would like to underline that, in general, the scalarized problems are not concave maximization problems, therefore the duality approach of convex analysis does not apply to them; instead, a new dual construction is proposed which is inspired by Toland's dualization (see [17, 19]) and guarantees a zero duality gap.

The paper is organized as follows. In Section 2, a special class of monotonic functions is studied with help of support functions and polar sets. In Section 3, a dual construction is proposed and duality relations between a scalarized problem and its dual are established. Section 4 is devoted to convergence of scalarizing functions and solution sets. In Section 5, an application to solve a concave maximization problem in a finite dimensional space is given. A new algorithm for generating weak efficient solutions is developed, its convergence is proved and some small size numerical examples are presented which show the applicability of the method.

2 Polar sets and monotonic functions

Given a nonempty set $A \subseteq Y$, its support function s_A is defined on Y' by

$$s_A(\xi) = \sup_{y\in A} \langle \xi, y\rangle \quad \text{for} \quad \xi \in Y';$$

and the polar set of A is a subset $A^\circ \subseteq Y'$ defined by

$$A^\circ = \{\xi \in Y' : s_A(\xi) \leq 1\} \, .$$

Similarly, the support function of a nonempty set $M \subseteq Y'$ is a function s_M defined on Y and the polar set of M is a subset $M^\circ \subseteq Y$.

Let ϕ: $Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function and let $D \subseteq Y$. We say that ϕ is nondecreasing on D if $y_1 = y_2 + c$ with $y_1, y_2 \in D$ and $c \in C$ implies $\phi(y_1) \geq \phi(y_2)$; and it is increasing on D if $y_1 = y_2 + c$ with $y_1, y_2 \in D$ and $c \in \text{int}C$ implies $\phi(y_1) > \phi(y_2)$.

Let $\mathcal{A}$ be the family of subsets A of Y which satisfy $A \cap C \neq \emptyset$. We shall adopt the following convention

$$\frac{r}{0} = \begin{cases} +\infty & \text{if } \; 0 < r \; \text{ or } \; r = +\infty, \\ -\infty & \text{if } \; 0 > r \; \text{ or } \; r = -\infty, \\ 0 & \text{if } \; r = 0. \end{cases}$$

Let Λ be a closed base of C^+, that is $C^+ = \{t\xi : \xi \in \Lambda \; , \; t \geq 0\}$ and $0 \notin \Lambda$. For every $A \in \mathcal{A}$ we define a function $g_A : Y \longrightarrow [0, \infty]$ by

$$g_A(y) = \sup_{\xi \in \Lambda} \frac{\langle \xi, y \rangle^+}{s_A(\xi)},$$

where $\langle \xi, y \rangle^+ = \max\{\langle \xi, y \rangle, 0\}$. This function will play a crucial role in solving problem (VP). Below we establish some of its properties.

Proposition 2.1 *Let $A \in \mathcal{A}$. Then the following assertions hold*:

(i) *g_A is nondecreasing, lower semicontinuous and sublinear on Y and coincides with the support function of the set $A^o \cap C^+$.*
(ii) *g_A is increasing on C provided that Λ is weakly compact and $+\infty > \sup_{\xi \in \Lambda} s_A(\xi) > 0$.*

Proof To prove (i), let $\xi \in \Lambda$ be given. Being the max function of two nondecreasing, continuous and linear functionals, the function $\langle \xi, . \rangle^+$ is nondecreasing, continuous and sublinear. If $s_A(\xi)$ is strictly positive, or equal to $+\infty$, then it is clear that the function $\frac{\langle \xi, . \rangle^+}{s_A(\xi)}$ is nondecreasing, continuous and sublinear. If $s_A(\xi) = 0$, it follows from our convention that $\frac{\langle \xi, . \rangle^+}{s_A(\xi)}$ is nondecreasing and sublinear too. This function is no longer continuous, but lower semicontinuous. Then the function $g_A(.)$, being the sup function of a family of nondecreasing, sublinear and lower semicontinuous functions, must share the same property. To prove that g_A and $s_{A^o \cap C^+}$ coincide, let $y \in Y$ be fixed. First we show that

$$g_A(y) \leq s_{A^o \cap C^+}(y) \; . \tag{2.1}$$

To this end, notice that since $A^o \cap C^+$ contains the origin, the support function on the right hand side of (2.1) is nonnegative. Therefore (2.1) will follow if we can show that

$$\frac{\langle \xi, y \rangle}{s_A(\xi)} \leq s_{A^o \cap C^+}(y) \tag{2.2}$$

for those vectors $\xi \in \Lambda$ for which $\langle \xi, y \rangle > 0$. Let ξ be such a vector. If $s_A(\xi) = 0$, then $t\xi \in A^\circ \cap C^+$ for each $t \geq 0$. Hence $s_{A^\circ \cap C^+}(y) \geq t\langle \xi, y \rangle$ for all $t > 0$. By this $s_{A^\circ \cap C^+}(y) = +\infty$ and (2.2) is obtained. If $s_A(\xi) > 0$, then $\frac{\xi}{s_A(\xi)}$ belongs to $A^\circ \cap C^+$ because $\xi \in \Lambda$ and $s_A(\frac{\xi}{s_A(\xi)}) = 1$. Then $\langle \frac{\xi}{s_A(\xi)}, y \rangle \leq s_{A^\circ \cap C^+}(y)$ which is nothing but (2.2).

Now we prove the converse inequality of (2.1). This will be done if we can show that for every $\zeta \in A^o \cap C^+$ one has inequality $\langle \zeta, y \rangle \leq g_A(y)$. Indeed, let $t \geq 0$ and $\xi \in \Lambda$ be such that $\zeta = t\xi$. Such t and ξ exist because λ is a base of C^+. If $\langle \zeta, y \rangle$ is nonpositive, then by definition $\langle \xi, y \rangle^+ = 0$ and $g_A(y) \geq 0$. If $\langle \zeta, y \rangle$ is positive, then $0 \leq s_A(\zeta) \leq 1$ and due to the homogeneity of s_A we derive

$$\langle \zeta, y \rangle \; \leq \frac{\langle \zeta, y \rangle}{s_A(\zeta)} = \frac{\langle \xi, y \rangle}{s_A(\xi)} \leq g_A(y).$$

Thus, equality holds in (2.1).

For the second assertion, let $y_1 = y_2 + c$ where $y_2 \in C$ and $c \in \text{int}C$. It follows from the hypothesis that

$$\inf_{\xi\in\Lambda} \frac{\langle\xi, c\rangle}{s_A(\xi)} \geq \frac{\inf_{\xi\in\Lambda} \langle\xi, c\rangle}{\sup_{\xi\in\Lambda} s_A(\xi)} > 0 .$$

Then we derive

$$g_A(y_1) = \sup_{\xi\in\Lambda} \frac{\langle\xi, y_2 + c\rangle}{s_A(\xi)} \geq \sup_{\xi\in\Lambda} \frac{\langle\xi, y_2\rangle^+}{s_A(\xi)} + \inf_{\xi\in\Lambda} \frac{\langle\xi, c\rangle}{s_A(\xi)} > g_A(y_2),$$

as requested. □

In the sequel, $\text{co}(A)$ denotes the convex hull of A and $\overline{\text{co}}(A)$ denotes its closed convex hull. We also use the notation $\text{cl}(A)$ for the closure of A.

Lemma 2.2 *For every $A \in \mathcal{A}$ one has*

(i) $\overline{\text{co}}(A^{\circ\circ} \cup (-C)) = \overline{\text{co}}(A \cup (-C)) = \overline{\text{co}}(A - C) = \text{cl}(A^{\circ\circ} - C)$.
(ii) $(A^\circ \cap C^+)^\circ = \text{cl}(A^{\circ\circ} - C)$.

Proof We observe first that $A^\circ \cap C^+ = (A \cup (-C))^\circ$. Hence

$$\overline{\text{co}}(A \cup (-C)) = (A \cup (-C))^{\circ\circ} = (A^\circ \cap C^+)^\circ = \overline{\text{co}}(A^{\circ\circ} \cup (-C)) . \quad (2.3)$$

Furthermore, as $0 \in C$ and $A \cap C \neq \phi$, one has $A \cup (-C) \subseteq A - C$ which together with the inclusion $A \subseteq A^{\circ\circ}$ implies

$$\overline{\text{co}}(A \cup (-C)) \subseteq \overline{\text{co}}(A - C) \subseteq \text{cl}(A^{\circ\circ} - C). \quad (2.4)$$

On the other hand, $A^{\circ\circ}$ being a convex set that contains the origin and C being a convex cone, we derive for each $a \in A^{\circ\circ}$ and $y \in -C$ that

$$a + y = \lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right) a + \frac{1}{n}(ny)\right) \in \overline{\text{co}}(A^{\circ\circ} \cup (-C)) .$$

This shows that $\text{cl}(A^{\circ\circ} - C) \subseteq \overline{\text{co}}(A^{\circ\circ} \cup (-C))$. Combine this inclusion with (2.3) and (2.4) to deduce (i).

To prove (ii) it suffices to use (i) and to observe the equality

$$(A^\circ \cap C^+)^\circ = \overline{\text{co}}(A \cup (-C)),$$

which is true for any $A \subseteq Y$. □

Proposition 2.3 *For any subsets $A_1, A_2 \in \mathcal{A}$, the following assertions are equivalent*

(i) $g_{A_1} = g_{A_2}$.
(ii) $\overline{\text{co}}(A_1 - C) = \overline{\text{co}}(A_2 - C)$.
(iii) $A_1^\circ \cap C^+ = A_2^\circ \cap C^+$.
(iv) $\text{cl}(A_1^{\circ\circ} - C) = cl(A_2^{\circ\circ} - C)$.

Proof Let us denote by $\sigma(Y', Y)$ the weakest locally convex topology on Y' for which all the linear functionals $\langle y, . \rangle$, $y \in Y$, are continuous. Then, the equivalence between (i) and (iii) follows from Proposition 2.1 (i) and from the fact that two convex sets, that are closed with respect to $\sigma(Y', Y)$, coincide if and only if their support functions coincide (see [7]). The equivalence between (ii) and (iv) is derived from Lemma 2.2. Now we prove the implication (ii) $\Longrightarrow$ (i). For this, it suffices to observe that for $\xi \in \Lambda$ one has

$$s_{A_i}(\xi) = s_{\overline{\mathrm{co}}(A_i - C)}(\xi), \quad i = 1, 2.$$

Finally, the implication (iii) $\Longrightarrow$ (iv) is obtained from Lemma 2.2 (ii). □

Corollary 2.4 *Assume that $A \in \mathcal{A}$ and $A \subseteq \Omega \subseteq Y$. Then*

$$g_A = g_{A-C} = g_{\Omega \cap (A-C)} = g_{\overline{\mathrm{co}}(A-C)} = g_{\Omega \cap \, \overline{\mathrm{co}}(A-C)}.$$

Proof Since $s_A(\xi) = s_{A-C}(\xi) = s_{\overline{\mathrm{co}}(A-C)}(\xi)$ for each $\xi \in \Lambda$, one has $g_A = g_{A-C} = g_{\overline{\mathrm{co}}(A-C)}$. Furthermore, the inclusions $A \subseteq \Omega \cap (A - C) \subset A - C$ imply $A^\circ \supseteq (\Omega \cap (A - C))^\circ \supseteq (A - C)^\circ$. The latter inclusions and Proposition 2.1 (i) yield

$$g_A(y) \geq g_{\Omega \cap (A-C)}(y) \geq g_{A-C}(y), \quad \forall y \in Y$$

and equality follows. □

Corollary 2.5 *Let $A_1, A_2 \in \mathcal{A}$ and $A_1, A_2 \subseteq \Omega \subseteq Y$. The following assertions are equivalent*

(i) $g_{A_1} = g_{A_2}$.
(ii) $\Omega \cap \, \overline{\mathrm{co}}(A_1 - C) = \Omega \cap \, \overline{\mathrm{co}}(A_2 - C)$.
If in addition A_1 and A_2 are convex compact, then the above is equivalent to
(iii) $A_1 - C = A_2 - C$.

Proof The implication (i) $\Longrightarrow$ (ii) follows from Proposition 2.3, while the implication (ii) $\Longrightarrow$ (i) is obtained from Corollary 2.4. The last part of the corollary is immediate because $A_1 - C$ and $A_2 - C$ are convex closed. □

We recall that the nondecreasing hull of u: $Y \to \mathbb{R}$ is the function $y \mapsto \tilde{u}(y) := \inf\{u(x)$: $x \in y + C\}$, which actually coincides with the infimal convolution of u and of the indicator function i_{-C} of the set $-C$, namely $\tilde{u} = u \Box i_C$.

Proposition 2.6 *Assume that Y is reflexive and that $A \in \mathcal{A}$ is bounded. Then g_A coincides with the nondecreasing hull of the support function s_{A°.*

Proof Since A is bounded, one has $0 \in \mathrm{int}\ A^\circ$. Then

$$g_A = s_{A^\circ \cap C^+} = (i_{A^\circ} + i_{C^+})^* = s_{A^\circ} \,\Box\, i_{-C} \,.$$

Here $(i_{A^\circ} + i_{C^+})^*$ is the conjugate of two convex functions and the last equality is obtained due to the condition dom $i_{C^+} \cap$ int(dom i_{A°) $\neq \emptyset$. □

3 Duality

Throughout this section we shall make the following assumption:

$$f(X) \text{ is a bounded set with } f(X) \cap \text{int}C \neq \emptyset. \tag{H}$$

This assumption implies the existence of some $\xi_* \in \Lambda$ such that $s_{f(X)}(\xi_*) > 0$. Denote by $\mathcal{A}_0 \subseteq \mathcal{A}$ the family of nonempty bounded subsets $A \subseteq Y$ satisfying

$$f(X) \subseteq A \subseteq \{\xi_*/s_{f(X)}(\xi_*)\}^\circ.$$

For each $A \in \mathcal{A}_0$, consider the following problem called a scalarized problem of (VP) (see [8, 10, 11, 20] for other kinds of scalarized problems):

$$\begin{aligned} &\max \sup_{\xi\in\Lambda} \frac{\langle \xi, f(x)\rangle}{s_A(\xi)}, \\ &\text{s.t.} \quad x \in X. \end{aligned} \tag{P_A}$$

The following result shows the importance of this particular scalarization.

Theorem 3.1 *Under hypothesis* (*H*) *one has*

(i) *the optimal value of problem* (P_A) *is equal to 1*;
(ii) *every optimal solution of* (P_A) *is a weak efficient solution of* (*VP*) *provided that* Λ *is weakly compact and that* $0 < \sup_{\xi\in\Lambda} s_A(\xi) < +\infty$;
(iii) *every weak efficient solution of* (*VP*) *is an optimal solution of* (P_A) *provided* $A = f(X)$ *and* $f(X) - C$ *is a convex set.*

Proof Let us denote $v(P_A) = \sup_{x\in X,\xi\in\Lambda} \frac{\langle \xi, f(x)\rangle}{s_A(\xi)}$, which in fact is the optimal value of (P_A). We note that $f(X) \subseteq A$ implies $s_{f(X)}(\xi) \le s_A(\xi)$ for each $\xi \in C^+$ and by this $v(P_A) \le 1$. On the other hand, $A \subseteq \left\{\frac{\xi_*}{s_{f(X)}(\xi_*)}\right\}^\circ$ yields $s_A(\xi_*) \le s_{f(X)}(\xi_*)$ and therefore $v(P_A) \ge 1$. Consequently, equality $v(P_A) = 1$ is obtained.

Furthermore, by an argument similar to that in the proof of Proposition 2.1, one verifies easily that the function $y \mapsto \sup_{\xi\in\Lambda} \frac{\langle \xi, y\rangle}{s_A(\xi)}$ is increasing on Y. Consequently, optimal solutions of (P_A) are weak efficient solutions of (VP).

For the last assertion, let $x_0 \in X$ be a weak efficient solution of (VP). Then $(f(X) - C) \cap (f(x_0) + \text{int}\, C) = \emptyset$. Separating these convex sets we find some nonzero vector $\xi \in Y'$ such that

$$\langle \xi, f(x)\rangle < \langle \xi, f(x_0) + c\rangle \quad \text{for all} \quad x \in X \quad \text{and} \quad c \in \text{int}\, C.$$

This implies, in particular, that $\xi \in C^+ \setminus \{0\}$ and so we may assume that $\xi \in \Lambda$. Moreover, $\langle \xi, f(x_0)\rangle = s_A(\xi)$. Therefore, in view of (i) the point x_0 is an optimal solution of (P_A). □

We notice that the convexity required in the above theorem is satisfied when X is a convex set and f is concave in the sense that $f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_1) \in C$ for any $t \in [0, 1]$ and for all $x_1, x_2 \in X$. It is known that most useful constructions of duality for vector optimization problems lead to problems whose data are set-valued functions (see [12, 13]). To avoid this complication, we find a dual for the scalarized problem (P_A) instead. The main difficulty is that even when the set X is convex and the function f is concave, (P_A) is no longer a concave maximization problem. Therefore, the usual Fenchel–Moreau–Rockafellar duality approach of convex analysis is not suitable to our case. The construction below is much inspired by the approach of Toland's dualization (see [17, 19] related to the duality in optimization of the difference of convex functions). First observe that by hypothesis (H) problem (P_A) can be written as

$$\begin{aligned} &\max g_A \circ f(x), \\ &\text{s.t.} \quad x \in X, \end{aligned} \tag{P'_A}$$

which, in view of Proposition 2.1, is expressed in the form

$$\begin{aligned} &\max s_{A^\circ \cap C^+} \circ f(x), \\ &\text{s.t.} \quad x \in X. \end{aligned}$$

We exchange the suprema in this latter problem to obtain the dual of (P_A):

$$\begin{aligned} &\max s_{f(X)}(\xi), \\ &\text{s.t.} \quad \xi \in A^\circ \cap C^+. \end{aligned} \tag{Q_A}$$

It is clear that there is no gap between the optimal values of (P_A) and (Q_A), which is a common feature of Toland type duality. Thus, in view of Theorem 3.1, the optimal value $v(Q_A)$ of the dual problem is equal to 1 too. The question that remains is how optimal solutions of (P_A) are linked with optimal solutions of (Q_A). Before tackling this question, let us give an example to illustrate the construction of a scalarized problem and its dual. Let $Y = \mathbb{R}^2,\ C = \mathbb{R}^2_+$, and let

$$X = \left\{(x_1, x_2) \in \mathbb{R}^2\colon x_1^2 + x_2^2 \le \tfrac{1}{4},\ x_1 \ge 0,\ x_2 \ge 0\right\},$$
$$A = \left\{(x_1, x_2) \in \mathbb{R}^2\colon 0 \le x_1 \le \tfrac{1}{2},\ 0 \le x_2 \le \tfrac{1}{2}\right\}.$$

Let $f\colon X \to \mathbb{R}^2$ be the identity function. The dual cone C^+ coincides with $\mathbb{R}^2_+$ and the standard simplex $\Lambda = \{(\xi_1, \xi_2) \in \mathbb{R}^2_+\colon \xi_1 + \xi_2 = 1\}$ can be used as a base of C^+. The support function s_A satisfies

$$s_A(\xi) = \tfrac{1}{2} \quad \text{for every } \xi \in \Lambda.$$

The problem (P_A) is written as

$$\begin{aligned} &\max\ \sup\nolimits_{(\xi_1,\xi_2)\in\Lambda} 2(\xi_1 x_1 + \xi_2 x_2), \\ &\text{s.t.} \quad (x_1, x_2) \in X, \end{aligned}$$

which is simplified as

$$\begin{aligned} &\max\ \max\{2x_1, 2x_2\}, \\ &\text{s.t.} \quad (x_1, x_2) \in X \end{aligned}$$

because $\xi_1 x_1 + \xi_2 x_2$ is a linear function of $\xi \in \Lambda$ for every fixed (x_1, x_2). In order to construct the problem (Q_A), let us compute the polar of A:

$$\begin{aligned} A^\circ &= \{(\xi_1, \xi_2) \in \mathbb{R}^2\colon \xi_1 x_1 + \xi_2 x_2 \le 1,\ \forall\ (x_1, x_2) \in A\} \\ &= \{(\xi_1, \xi_2) \in \mathbb{R}^2\colon \xi_1 + \xi_2 \le 2,\ \xi_1 \le 2,\ \xi_2 \le 2\}. \end{aligned}$$

The support function $s_{f(X)}$ is given by

$$s_{f(X)} = \sup_{x\in X}[\xi_1 x_1 + \xi_2 x_2] = \frac{\sqrt{\xi_1^2 + \xi_2^2}}{2}.$$

The dual problem (Q_A) is then written as

$$\begin{array}{l} \max \frac{\sqrt{\xi_1^2+\xi_2^2}}{2}, \\ \text{s.t.} \quad \xi_1 + \xi_2 \le 2,\ \xi_1 \le 2 \text{ and } \xi_2 \le 2. \end{array}$$

Observe that in this example, (P_A) is a maximization problem over a convex set X while (Q_A) is a maximization problem over a polytope of the dual space Y'.

As promised, let us now derive a relationship between solutions of problem (P_A) and those of (Q_A). Consider the following auxiliary problems

$$\begin{array}{l} \max \langle \xi,\ f(x)\rangle, \\ \text{s.t.} \quad \xi \in A^\circ \cap C^+ \end{array} \tag{P_x}$$

for a fixed $x \in X$, and

$$\begin{array}{l} \max \langle \xi,\ f(x)\rangle, \\ \text{s.t.} \quad x \in X \end{array} \tag{Q_ξ}$$

for a fixed $\xi \in A^\circ \cap C^+$.
Recall that the normal cone to a convex set $K \subseteq Y$ at $a \in K$ is defined by

$$N(K, a) = \{\xi \in Y' \colon \langle \xi, y - a\rangle \le 0 \quad \text{for every } y \in K\}\ .$$

Below are some optimality conditions for problem (P_x) and (Q_ξ). The notation ∂h stands for the convex subdifferential of h.

Proposition 3.2 *The following assertions hold:*

(i) $v(P_A) = \sup_{x\in X} v(P_x)$ *and* $v(Q_A) = \sup_{\xi\in A^\circ\cap C^+} v(Q_\xi)$.
(ii) $\xi \in A^\circ \cap C^+$ *is an optimal solution of* (P_x) *if and only if there is some* $y \in -C$ *such that* $\langle \xi, y\rangle = 0$ *and* $f(x) - y \in N(A^\circ, \xi)$.
(iii) *Assume that* X *is convex and* f *is concave and continuous. Then* $x \in X$ *is an optimal solution of* (Q_ξ) *if and only if*

$$-N(X, x) \cap \partial(-\xi \circ f)(x) \ne \emptyset\ .$$

Proof The first assertion is immediate from the definitions of the problems (P_A) and (Q_A). For the second assertion, we note that the problem (P_x) is a linear problem, so that $\xi \in A^\circ \cap C^+$ is an optimal solution if and only if $f(x) \in N(A^\circ \cap C^+, \xi)$. Since A is bounded, one has $0 \in \text{int}\, A^\circ$ and

$$N(A^\circ \cap C^+, \xi) = N(A^\circ, \xi) + N(C^+, \xi) = N(A^\circ, \xi) + (-C) \cap \{\xi\}^\perp,$$

where $\{\xi\}^\perp$ consists of vectors $a \in Y$ such that $\langle \xi, a\rangle = 0$. This proves (ii).
When f is a concave function, the problem (Q_ξ) is a concave maximization problem so that $x \in X$ is an optimal solution if and only if

$$0 \in \partial(-\langle \xi,\ f(.)\rangle + i_X(.))(x)\ .$$

Since f is continuous, we obtain

$$0 \in \partial(-\langle \xi,\ f(\cdot)\rangle)(x) + N(X, x)$$

which implies (iii). □

A duality relation between the optimal solutions of (P_A) and those of (Q_A) is given next.

Theorem 3.3 *The following assertions hold*

(i) *If* $x \in S(P_A)$ *and* $\xi \in S(P_x)$, *then* $\xi \in S(Q_A)$.
(ii) *If* $\xi \in S(Q_A)$ *and* $x \in S(Q_\xi)$, *then* $x \in S(P_A)$.

In both cases,

$$v(P_A) = v(P_x) = \langle f(x), \xi\rangle = v(Q_x) = v(Q_A)\ . \tag{3.1}$$

Proof Assume that $x \in S(P_A)$ and $\xi \in S(P_x)$. Then it is clear that $v(P_A) = v(P_x)$ and $v(P_A) = \langle f(x), \xi\rangle$ because $\xi \in S(P_x)$. Moreover,

$$\langle f(x), \xi\rangle \le v(Q_\xi) \le v(Q_A)\ .$$

According to Theorem 3.1, we deduce equalities (3.1) which show that $\xi \in S(Q_A)$. Assertion (ii) is proved in a similar way. □

The expressions of the solution sets $S(P_A)$ and S(VP) given in the two corollaries below are helpful in development of numerical methods for solving problem (VP).

Corollary 3.4 *Assume that either of the following conditions holds*:

(i) *A and C are polyhedral*;
(ii) $0 \in \mathrm{int}\overline{\mathrm{co}}(A \cup (-C))$.

Then one has

$$S(P_A) = \bigcup_{\xi\in S(Q_A)} S(Q_\xi) \ = \bigcup_{\xi\in S(Q_A)} \{x \in X : \langle f(x), \xi\rangle = 1\}.$$

Proof According to Theorem 3.3, it suffices to show that if $x \in S(P_A)$, there exists $\xi \in S(Q_A)$ such that $x \in S(Q_\xi)$. Indeed, consider the linear function $\xi \mapsto \langle \xi, f(x)\rangle$ on $A^\circ \cap C^+$. Under (i), the set $A^\circ \cap C^+$ is polyhedral, while under (ii) it is bounded. Thus, in both cases, there exists some $\xi \in A^\circ \cap C^+$ such that

$$\langle \xi, f(x)\rangle = v(P_x) = v(P_A) = v(Q_A) \ge v(Q_\xi) = \langle \xi, f(x)\rangle,$$

which shows that $x \in S(Q_\xi)$. The second equality follows from the first equality, Theorems 3.1 and 3.3. □

Corollary 3.5 *Assume that* Λ *is weakly compact and that* $f(X) \subseteq C$ *is bounded, closed with* $f(X) - C$ *being convex. Then*

$$S(\mathrm{VP}) = \bigcup_{\xi\in S(Q_{f(X)})} \{x \in X\colon \langle \xi, f(x)\rangle = 1\}.$$

Proof Invoke Theorem 3.1 and Corollary 3.4. □

4 Convergence

According to Theorem 3.1, in order to obtain all weak efficient solutions of the problem (VP) with help of the scalarized problem (P_A), we have to compute the function g_A with $A = f(X)$. A practical way to do it is to approximate $f(X)$ by a sequence of sets having simple structure, say a sequence of polytopes A_k, and then construct g_{A_k}. Our goal is to show that the functions g_{A_k} converge to g_A when A_k tends to the set $f(X)$ in a suitable sense. Throughout this section we assume that Λ is bounded and denote $\delta := \sup_{\xi\in\Lambda} ||\xi||$. Given two nonempty closed subsets $A_1, A_2 \subseteq Y$, the Hausdorff distance between them is defined by

$$h(A_1, A_2) = \inf\{t > 0\colon A_1 \subseteq A_2 + tB,\ \ A_2 \subseteq A_1 + tB\},$$

where B denotes the closed unit ball in Y. Let $\{A_n\}_{n=1}^{\infty} \subseteq Y$ be a sequence of nonempty closed sets. Its upper limit and lower limit in the sense of Kuratowski and Painlevé are defined as

$$\limsup_{n\to\infty} A_n := \left\{\lim_{i\to\infty} a_{n_i} : a_{n_i} \in A_{n_i}, i = 1, 2, \ldots\right\},$$
$$\liminf_{n\to\infty} A_n := \left\{\lim_{n\to\infty} a_n : a_n \in A_n, n = 1, 2, \ldots\right\}.$$

We say that this sequence H-converges to a closed set A if

$$\lim_{n\to\infty} h(A_n, A) = 0,$$

and KP-converges to A if

$$\limsup_{n\to\infty} A_n \subseteq A \subseteq \liminf_{n\to\infty} A_n.$$

For more on convergence of sets see [1]. For a nonempty set $A \subseteq Y$, together with the polar set A° we shall consider two other polar sets:

$$A^{-1} := \{\xi \in Y'\colon \langle\xi, a\rangle < -1 \quad \text{for all}\ \ a \in A\}$$
$$A^{-} := \{\xi \in Y'\colon \langle\xi, a\rangle \le 0 \quad \text{for all}\ \ a \in A\}.$$

It is evident that $A^{-1} \subseteq A^- \subseteq A^\circ$ and when A^{-1} is nonempty, the cone A^- is nontrivial and the set A° is unbounded. Direct verification gives the following formula to compute the polar set of the sum $A + \varepsilon B$. Let $A \subseteq Y$ be nonempty and let $\varepsilon > 0$, then the polar set $(A + \varepsilon B)^\circ$ is the set

$$\left\{\frac{\xi}{1+\varepsilon\|\xi\|} : \xi \in A^\circ\right\} \cup \left\{\frac{\xi}{\varepsilon\|\xi\| - 1} : \xi \in A^{-1}, \|\xi\| > \frac{1}{\varepsilon}\right\} \cup \left\{\xi\colon \xi \in A^-, \|\xi\| = \frac{1}{\varepsilon}\right\}.$$

Below are some elementary properties of polar sets that we shall need in the sequel.

Lemma 4.1 *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of closed sets in Y. The following assertions hold:*

(i) $\cap_{n=1}^{\infty} A_n^\circ = (\cup_{n=1}^{\infty} A_n)^\circ$.
(ii) $(\cap_{n=1}^{\infty} A_n)^\circ = \overline{\text{co}}(\cup_{n=1}^{\infty} A_n^\circ)$ *provided the sets A_n are convex closed and contain 0.*
(iii) *If $\{A_n\}_{n=1}^{\infty}$ H-converges to A, then for every positive number r, the sequence $\{A_n^\circ \cap (rB)\}_{n=1}^{\infty}$ H-converges to $A^\circ \cap (rB)$. If in addition $0 \in \text{int}A$, then $\{A_n^\circ\}_{n=1}^{\infty}$ H-converges to A°.*

Proof The two first assertions are known (see [7], p. 113). The last one is obtained from the formula to compute the polar set of the sum $A + \varepsilon B$, which we have presented before this lemma. □

Theorem 4.2 *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of closed sets H-converging to a closed set A with $0 \in \operatorname{int}(A - C)$. Then $\{g_{A_n}\}_{n=1}^{\infty}$ pointwise converges to g_A.*

Proof Observe first that the sequence $\{(A_n - C)\}_{n=1}^{\infty}$ H-converges to the set $A - C$. According to Lemma 4.1, the sequence of polar sets $\{(A_n - C)^\circ\}_{n=1}^{\infty}$ H-converges to $(A - C)^\circ$ too. Furthermore, as $(A_n - C)^\circ \subseteq C^+$, by applying Proposition 2.1 (i) and Corollary 2.5, we derive that $g_{A_n} = s_{(A_n - C)^\circ}$ and $g_A = s_{(A-C)^\circ}$. The boundedness of the set $(A - C)^\circ$ and the H-convergence of the sequence $\{(A_n - C)^\circ\}_{n=1}^{\infty}$ imply that $\{s_{(A_n-C)^\circ}(y)\}_{n=1}^{\infty}$ converges to $s_{(A-C)^\circ}(y)$ for every $y \in Y$. The proof is complete. □

This theorem can also be derived from the following estimate of the function g_A.

Lemma 4.3 *Assume that $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq A_2 + \varepsilon B$ for some $\varepsilon > 0$ and that*

$$e_{A_2} := \inf_{\xi \in \Lambda} s_{A_2}(\xi) > 0 .$$

Then for every $y \in Y$,

$$g_{A_1}(y) \geq g_{A_2}(y) - \frac{\varepsilon\delta^2 \|y\|}{(e_{A_2})^2}.$$

Proof We observe first that

$$s_{A_2+\varepsilon B}(\xi) = s_{A_2}(\xi) + \varepsilon\|\xi\| \quad \text{for each } \xi \in C^+ .$$

Hence for each $y \in Y$, one has

$$\begin{aligned}\frac{\langle \xi, y\rangle^+}{s_{A_1(\xi)}} &\geq \frac{\langle \xi, y\rangle^+}{s_{A_2+\varepsilon B}(\xi)} \geq \frac{\langle \xi, y\rangle^+}{s_{A_2}(\xi) + \varepsilon\|\xi\|} \\ &\geq \frac{\langle \xi, y\rangle^+}{s_{A_2}(\xi)} - \frac{\varepsilon\|\xi\|\langle \xi, y\rangle^+}{s_{A_2}(\xi)(s_{A_2}(\xi) + \varepsilon\|\xi\|)} \geq \frac{\langle \xi, y\rangle^+}{s_{A_2}(\xi)} - \frac{\varepsilon\delta^2\|y\|}{(e_{A_2})^2} .\end{aligned}$$

This implies

$$g_{A_1}(y) \geq g_{A_2}(y) - \frac{\varepsilon\delta^2 \|y\|}{(e_{A_2})^2}$$

as requested. □

Corollary 4.4 *Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of closed sets H-converging to a closed set A with $e_A > 0$.Then $\{g_{A_n}\}_{n=1}^{\infty}$ pointwise converges to g_A.*

Proof We observe that for each $\varepsilon > 0$, there is some $n_0 \geq 1$ such that $A_n \subseteq A + \varepsilon B$ and $A \subseteq A_n + \varepsilon B$ for $n \geq n_0$. According to Lemma 4.3, we derive

$$g_{A_n}(y) \geq g_A(y) - \frac{\varepsilon\delta\|y\|}{(e_A)^2} \geq g_{A_n}(y) - \varepsilon\delta\|y\| \left(\frac{1}{(e_{A_n})^2} + \frac{1}{(e_A)^2}\right) .$$

It is clear that $e_{A_n} \longrightarrow e_A$ as $n \longrightarrow \infty$. Hence $\lim_{n\to\infty} g_{A_n}(y) = g_A(y)$. □

It is worthwhile noticing that $e_A > 0$ if and only if $0 \in \operatorname{int}(\overline{\operatorname{co}}(A - C))$. This and Corollary 2.5 show that Theorem 4.2 and Corollary 4.4 can be derived from each other. We obtain a convergence property for solutions of scalarized problems and of dual problems.

Corollary 4.5 *Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of closed sets H-converging to a closed set A with $0 \in \operatorname{int}(A - C)$. Then*

$$\limsup_{n\to\infty} S(P_{A_n}) \subset S(P_A),$$
$$\limsup_{n\to\infty} S(Q_{A_n}) \subset S(Q_A).$$

Proof This follows immediately from Theorem 4.2. □

We now study the convergence of ε-solutions of scalarized problems. Given $\varepsilon > 0$, we say that $x_0 \in X$ is an ε-solution of problem (P'_A) if

$$g_A \circ f(x_0) + \varepsilon \geq g_A \circ f(x) \quad \text{for every } x \in X.$$

The set of all ε-solutions of (P'_A) (hence of (P_A) as well) is denoted by $S_\varepsilon(P'_A)$.

Proposition 4.6 *Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of nonempty, closed sets, which H-converges to a closed set A with $0 \in \operatorname{int}(A - C)$ and let X be compact. Then*

$$S(P_A) \subseteq \cap_{\varepsilon>0} \liminf_{n\to\infty} S_\varepsilon(P_{A_n}).$$

If in addition the sequence $\{A_n\}_{n=1}^{\infty}$ is monotone (either increasing or decreasing by inclusions), then for every $\varepsilon > 0$ there is some $n_0 > 0$ such that

$$S(P_A) \subseteq S_\varepsilon(P_{A_n}) \text{ for all } n \geq n_0$$

and in particular

$$\lim_{n\to\infty,\varepsilon\downarrow 0} h(S_\varepsilon(P_{A_n}), S(P_A)) = 0.$$

Proof The first inclusion is evident. For the second part of the proposition, assume that the sequence $\{A_n\}_{n=1}^{\infty}$ is monotone. Then the sequence $\{A_n^\circ \cap C^+\}_{n=1}^{\infty}$ is monotone too. By Proposition 2.1 (i), the sequence of scalarizing functions $\{g_{A_n}\}_{n=1}^{\infty}$ is monotone. In view of Theorem 4.2, this latter sequence pointwise converges to the continuous function g_A. By hypothesis, the set X is compact, we deduce that the sequence $\{g_{A_n} \circ f\}_{n=1}^{\infty}$ uniformly converges to $g_A \circ f$ on X. Now, let $\varepsilon > 0$ be given. As the set $S(P_A)$ is compact, there is $n_0 > 0$ such that $|g_{A_n} \circ f(x) - g_A \circ f(x)| < \varepsilon$ for every $n \geq n_0$ and for every $x \in S(P_A)$. Using Theorem 3.1, we derive

$$g_{A_n} \circ f(x) \geq g_A \circ f(x) - \varepsilon \geq 1 - \varepsilon,$$

which means that $x \in S_\varepsilon(P_{A_n})$ for $n \geq n_0$ and $x \in S(P_A)$. This in particular implies

$$S(P_A) \subseteq \liminf_{n\to\infty,\varepsilon\downarrow 0} S_\varepsilon(P_{A_n}).$$

We now show that

$$\limsup_{n\to\infty,\varepsilon\downarrow 0} S_\varepsilon(P_{A_n}) \subseteq S(P_A).$$

Let $x = \lim_{n_i\to\infty,\varepsilon_i\downarrow 0} x_{n_i,\varepsilon_i}$ where $x_{n_i,\varepsilon_i} \in S_{\varepsilon_i}(P_{A_{n_i}})$. Then in view of Theorem 3.1, we have

$$g_{A_{n_i}} \circ f(x_{n_i,\varepsilon_i}) \geq 1 - \varepsilon_i \quad \text{for all } i \geq 1.$$

By the uniform convergence of the sequence $\{g_{A_n} \circ f\}_{n=1}^{\infty}$ to $g_A \circ f$ on X we conclude $g_A \circ f(x) \geq 1$. Again by Theorem 3.1, $x \in S(P_A)$. In this way, the generalized sequence $\{S_\varepsilon(P_{A_n})\}_{n\geq 1,\varepsilon>0}$ KP-converges to $S(P_A)$. These sets being compact, we obtain H-convergence as well. □

5 Generating the solution set of problem (VP)

In this section we shall apply the analysis developed in the previous sections to solve a concave maximization problem in a finite dimensional space. Let us consider the problem:

$$\begin{array}{l}\text{WMax } f(x),\\ \text{s.t.} \quad x \in X,\end{array} \tag{VP}$$

where X is a nonempty subset of $\mathbb{R}^n$, $f = (f_1, \ldots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\mathbb{R}^m$ is equipped with the positive octant cone $\mathbb{R}^m_+$. We are interested in finding all weak efficient solutions of the problem, namely the set

$$S(\text{VP}) = \{x \in X\colon (f(X) - f(x)) \cap \text{ int } \mathbb{R}^m_+ = \emptyset\} .$$

Given $\varepsilon > 0$, we say that $x_0 \in X$ is an ε-solution of (VP) if

$$f(X) \cap ([f(x_0) + (\varepsilon, \ldots, \varepsilon)] + \text{ int } \mathbb{R}^m_+) = \emptyset .$$

The set of all ε-solutions of (VP) is denoted by $S_\varepsilon(\text{VP})$. The standard simplex

$$\Lambda = \left\{\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+\colon \sum_{i=1}^m \xi_i = 1\right\}$$

will serve as a base of the nonnegative polar cone $\mathbb{R}^m_+$. We shall make use of the notation $A^\diamond = (A - \mathbb{R}^m_+) \cap \mathbb{R}^m_+$ for $A \subseteq \mathbb{R}^m_+$ and assume throughout the following hypothesis:

$$f(X) \subset \text{int}\mathbb{R}^m_+ \text{ and the set } [f(X)]^\diamond \text{ is nonempty, compact and convex.} \tag{H$'$}$$

This hypothesis implies (H) and is fulfilled, for instance, when X is nonempty compact and convex, f is continuous and concave. By setting $A = [f(X)]^\diamond$ we have $f(X) \subseteq A$ and $A^\diamond = A$. Let $\overline{g}_A$ be a function defined on $\mathbb{R}^m$ by

$$\overline{g}_A(y) = \sup_{\xi \in \Lambda} \frac{\langle \xi, y\rangle}{s_A(\xi)} .$$

It is clear that $g_A(y) = \overline{g}_A(y)$ for each $y \in \mathbb{R}^m_+$. Moreover, $\overline{g}_A$ is continuous, sublinear and increasing (see the proof of Theorem 3.1).

Lemma 5.1 *Under the hypothesis* (H$'$) *and with* $A = [f(X)]^\diamond$ *one has* $S(P_A) = S(\text{VP})$. *Moreover, for* $\varepsilon \geq 0$, $x_0 \in X$ *is an* ε*-solution of* (VP) *if and only if there is some vector* $\xi \in \Lambda$ *such that* x_0 *is an* ε*-solution of* (Q_ξ).

Proof That the sets $S(P_A)$ and $S(\text{VP})$ coincide is immediate from Theorem 3.1. Now, let x_0 be an ε-solution of (VP). Then the set $f(X) - \mathbb{R}^m_+$ which is convex according to the hypothesis (H$'$) does not meet the convex set $f(x_0) + (\varepsilon, \ldots, \varepsilon) + \text{int}\mathbb{R}^m_+$. Separating them, we find some $\xi \in \Lambda$ such that

$$\langle \xi, f(x)\rangle \leq \langle \xi, f(x_0)\rangle + \varepsilon$$

for every $x \in X$. This shows that x_0 is an ε-solution of (Q_ξ). Conversely, if x_0 is not an ε-solution of (VP), then there are some $x \in X$ and $c \in \text{int}\mathbb{R}^m_+$ such that $f(x) = f(x_0) + (\varepsilon, \ldots, \varepsilon) + c$. We derive for each $\xi \in \Lambda$ that

$$\langle \xi, f(x)\rangle \geq \langle \xi, f(x_0)\rangle + \varepsilon + \langle \xi, c\rangle > \langle \xi, f(x_0)\rangle + \varepsilon.$$

Hence x_0 cannot be an ε-solution of (Q_ξ). The proof is complete. □

Proposition 5.2 *Under the hypothesis* (H'), *one has*

$$S(\mathrm{VP}) = \bigcup_{\xi \in bd_+(A^\circ)} S(Q_\xi) = \bigcup_{\xi \in bd_+(A^\circ)} \{x \in X : \langle \xi, f(x)\rangle = 1\},$$

where $bd_+(A^\circ)$ *denotes the intersection of the boundary of the set* A° *with the octant* $\mathbb{R}^m_+$. *In particular, if* A *is a polytope and* Γ *is a set of those vertices of* A° *that lie in* $\mathbb{R}^m_+$, *then*

$$S(\mathrm{VP}) = \bigcup_{\xi \in \Gamma} \{x \in X : \langle \xi, f(x)\rangle = 1\}.$$

Proof Since the solution set of problem (Q_A) is included in $bd_+(A^\circ)$, Corollary 3.4 yields

$$S(P_A) \subset \bigcup_{\xi \in bd_+(A^\circ)} S(Q_\xi) \ .$$

Moreover, for each nonnegative vector $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ the function $\langle \xi, \cdot\rangle$ is increasing, therefore one has

$$S(Q_\xi) \subseteq S(\mathrm{VP}) \ .$$

The first equality follows now from Lemma 5.1. The second equality is derived from Theorem 3.1. The equality of the particular case when A is a polytope is then evident. □

In view of the above proposition, for generating the solution set of (VP) it suffices to determine A° or more precisely $bd_+(A^\circ)$ and then solve the scalar problems (Q_ξ). In what follows we present an algorithm to solve problem (VP) by approximating the set A° from outside. The idea is to start up with a polyhedron $S_1 \supset A^\circ$ and to built up a sequence of polyhedra

$$S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots \supset A^\circ.$$

This can be done by the dual relation between a polyhedron and its polar (see e.g. [6]) which states that if two full-dimensional polyhedra P and S containing 0 are polar to each other, then there exists a 1-1 correspondence between the set of facets of P not containing 0 and the set of nonzero vertices of S.

Denote by q the vector $(q_1, \ldots, q_m)$, where $q_1, \ldots, q_m$ are the optimal values of the following problems

$$\begin{aligned} &\max \ f_i(x), \\ &\text{s.t. } x \in X. \end{aligned}$$

It is easy to see that the vectors $y^1 = (q_1, \ldots, 0), \ldots, y^m = (0, \ldots, q_m)$ belong to $[f(X)]^\diamond$. Set

$$\begin{aligned} B_1 &= \mathrm{co}\{0, y^1, \ldots, y^m\}, \\ \hat{B}_1 &= B_1 - \mathbb{R}^m_+, \\ S_1 &= \{\xi \in \mathbb{R}^m : \langle \xi, y^i\rangle \le 1, \ i = 1, \ldots, m\} \cap \mathbb{R}^m_+, \\ \hat{S}_1 &= S_1 - \mathbb{R}^m_+. \end{aligned}$$

Denote by V_1 the vertex set of S_1 which actually consists of the vectors

$$v = (v_1, \ldots, v_m) \quad \text{where} \quad v_i \in \left\{0, \frac{1}{q_i}\right\}, \quad i = 1, \ldots, m.$$

We construct S_{k+1} by induction. Assume that S_k is known together with its vertex set V_k. Define

$$V_k^* = \{v \in V_k \colon s_A(v) > 1\}.$$

If $V_k^* = \emptyset$, we set $S_{k+1} = S_k$. Otherwise define

$$S_{k+1} = S_k \cap \{\xi \in \mathbb{R}^m \colon \langle \xi, y_v \rangle \le 1, \quad v \in V_k^*\},$$

where y_v is a maximum of the linear function $\langle v, . \rangle$ on A and

$$\begin{aligned}\hat{S}_{k+1} &= S_{k+1} - \mathbb{R}^m_+,\\ B_{k+1} &= \mathrm{co}(B_k \cup \{y_v : v \in V_k^*\}),\\ \hat{B}_{k+1} &= B_{k+1} - \mathbb{R}^m_+.\end{aligned}$$

Lemma 5.3 *For every $k \ge 1$ the sets B_k and $\hat{B}_k$ are polar to $\hat{S}_k$ and S_k respectively.*

Proof We prove this lemma by induction on k. For $k = 1$, the conclusion is immediate from Lemma 2.2. Assume that $(B_k)^\circ = \hat{S}_k$ and $(\hat{B}_k)^\circ = S_k$ for some $k \ge 1$. Notice that B_k, $\hat{B}_k$, S_k and $\hat{S}_k$ are convex polyhedra which contain the origin of the space. By polarity and by induction we obtain

$$\begin{aligned}(S_{k+1})^\circ &= (S_k \cap \{\xi \in \mathbb{R}^m \colon \langle \xi, y_v \rangle \le 1, v \in V_k^*\})^\circ,\\ &= \overline{\mathrm{co}}(S_k^\circ \cup \{\xi \in \mathbb{R}^m \colon \langle \xi, y_v \rangle \le 1, v \in V_k^*\}^\circ),\\ &= \overline{\mathrm{co}}(\hat{B}_k \cup \{y_v \colon v \in V_k^*\}),\\ &= \overline{\mathrm{co}}((B_k - \mathbb{R}^m_+) \cup \{y_v \colon v \in V_k^*\}),\\ &= \overline{\mathrm{co}}[B_k \cup \{y_v \colon v \in V_k^*\}] - \mathbb{R}^m_+,\\ &= \mathrm{co}[B_k \cup \{y_v \colon v \in V_k^*\}] - \mathbb{R}^m_+,\\ &= B_{k+1} - \mathbb{R}^m_+ = \hat{B}_{k+1}.\end{aligned}$$

This implies $(\hat{S}_{k+1})^\circ = (S_{k+1} - \mathbb{R}^m_+)^\circ = (S_{k+1})^\circ \cap \mathbb{R}^m_+ = \hat{B}_{k+1} \cap \mathbb{R}^m_+ = B_{k+1}$. The proof is complete. □

Theorem 5.4 *Under the hypothesis (H), the following assertions hold:*

(i) $S_{k+1} \subseteq S_k$.
(ii) $A^\circ \cap \mathbb{R}^m_+ \subseteq S_k$ *and* $A^\circ = (A^\circ \cap \mathbb{R}^m_+) - \mathbb{R}^m_+ \subseteq \hat{S}_k$.
(iii) *If* $V_k^* = \emptyset$ *for some* $k \ge 1$, *then* $\hat{S}_k = A^\circ$.
(iv) *The sequence* $\{\hat{S}_k\}_{k=1}^\infty$ *H-converges to* A°.

Proof The first assertion is evident from the definition. We prove the first inclusion of (ii) by induction. For $k = 1$, we have $B_1 \subseteq A$ which implies $A^\circ \subseteq (B_1)^\circ = \hat{S}_1 = S_1 - \mathbb{R}^m_+$ by Lemma 5.3. Hence

$$A^\circ \cap \mathbb{R}^m_+ \subseteq (S_1 - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ = S_1.$$

Furthermore, by the definition one has

$$A = [f(X)]^\diamond = (f(X) - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ = \overline{\mathrm{co}}(A \cup \{0\}) = A^{\circ\circ}.$$

This yields

$$(A^\circ \cap \mathbb{R}^m_+)^\circ = A^{\circ\circ} - \mathbb{R}^m_+ = A - \mathbb{R}^m_+,$$

which in view of Lemma 2.2 gives

$$(A^\circ \cap \mathbb{R}^m_+) - \mathbb{R}^m_+ = \{(A^\circ \cap \mathbb{R}^m_+)^\circ \cap \mathbb{R}^m_+\}^\circ = A^\circ.$$

The latter equalities and the first inclusion of (ii) show that $A^\circ \subseteq \hat{S}_k$.

We now assume $V_k^* = \emptyset$. This means that $V_k \subseteq A^\circ$. Since $V_k \subseteq \mathbb{R}^m_+$, one has $V_k \subseteq A^\circ \cap \mathbb{R}^m_+$ which implies that $S_k \subseteq A^\circ \cap \mathbb{R}^m_+$. In view of (ii) this latter inclusion becomes equality and hence

$$A^\circ = A^\circ \cap \mathbb{R}^m_+ - \mathbb{R}^m_+ = S_k - \mathbb{R}^m_+ = \hat{S}_k.$$

To prove (iv), first we show that the sequence $\{B_k\}_{k=1}^\infty$ H-converges to A. Indeed, let $y \in A \setminus B_k$. According to Lemma 5.3, there exist some $v \in \hat{S}_k$ such that $\langle v, y\rangle > 1$. Since $\hat{S}_k = S_k - \mathbb{R}^m_+$, there are $v_1, \ldots, v_l \in V_k$, $c \in \mathbb{R}^m_+$ and $\lambda_1, \ldots, \lambda_l \geq 0$ with $\sum_{i=1}^l \lambda_i = 1$ such that $v = \sum_{i=1}^l \lambda_i v_i - c$. Then

$$\sum_{i=1}^l \lambda_i \langle v_i, y\rangle - \langle c, y\rangle > 1.$$

The vector y being positive, the value $\langle c, y\rangle$ is nonnegative. It follows from the above inequality that

$$\langle v_i, y\rangle > 1 \quad \text{for some} \quad i \in \{1, \ldots, l\}.$$

Thus, for $y \in A \setminus B_k$, there exists $v_i \in V_k$ such that $\langle v_i, y\rangle > 1$. As the power of V_k is finite we find some $v_0 \in V_k$ and $y' \in B_k$ such that

$$\langle v_0, y'\rangle = 1, \quad \langle v_0, y\rangle > 1, \quad y = \langle v_0, y\rangle y'.$$

Let us estimate the distance from y to B_k:

$$\begin{aligned} \min_{z \in B_k} \|y - z\| &\leq \|y - y'\| \\ &\leq \|y'\|(\langle v_0, y\rangle - 1) \\ &\leq \big(\max_{a \in A} \|a\|\big)\big(\max_{v \in V_k^*} s_A(v) - 1\big). \end{aligned}$$

Since this holds for any $y \in A \setminus B_k$ one deduces that

$$h(B_k, A) \leq \big(\max_{a \in A} \|a\|\big)\big(\max_{v \in V_k^*} s_A(v) - 1\big).$$

On the other hand, for each $v \in V_k^*$ one has

$$\langle v, y_v\rangle = s_A(v) > 1. \tag{5.1}$$

Let $v_k \in V_k^*$ be such that $\max_{v \in V_k^*} s_A(v) = s_A(v_k) = \langle v_k, y_{v_k}\rangle$. Since the sequences $\{v_k\}_{k=1}^\infty$ and $\{y_{v_k}\}_{k=1}^\infty$ are bounded, without loss of generality one may assume that they converge to $\bar{v}$ and $\bar{y}$ respectively. It follows from (5.1) that $\langle \bar{v}, \bar{y}\rangle \geq 1$. By the construction of S_{k+1}, one has $\langle v_{k+1}, y_{v_k}\rangle \leq 1$, therefore $\langle \bar{v}, \bar{y}\rangle \leq 1$. As a result

$$\lim_{k \to \infty} \max_{v \in V_k^*} s_A(v) = \langle \bar{v}, \bar{y}\rangle = 1,$$

which implies that $\{B_k\}_{k=1}^\infty$ H-converges to A and the assertion (iv) follows from Lemma 4.1. □

5.1 Algorithm

Now we are in the position to present a new algorithm for solving problem (VP).

Step 1. *(Initialization).* Choose a small $\varepsilon > 0$. Find q_i as before and $b_i \in \mathbb{R}$ such that $q_i \geq f_i(x) > b_i$ for all $x \in X$ and $i = 1, \ldots, m$. Set $k = 1$ and for $i = 1, \ldots, m$:

$$f_i(x) = f_i(x) - b_i, \quad q_i = a_i - b_i, \quad y^i = q_i e_i,$$

where $e_1 = (1, \ldots, 0), \ldots, e_m = (0, \ldots, 1)$. Define

$$S_k = \{\xi \in \mathbb{R}^m_+ : \langle \xi, y^i \rangle \leq 1, i = 1, \ldots, m\}$$

and set $V_k = \{v = (v_1, \ldots, v_m) : v_i \in \left\{0, \frac{1}{q_i}\right\}, i = 1, \ldots, m\}$.

Step 2. For each $v \in V_k$ solve problem (P_v):

$$\max \ \langle v, y \rangle, \qquad \text{s.t.} \quad y \in [f(X)]^{\diamond}.$$

to obtain $s_A(v)$ and an optimal solution y_v.

Step 3. Set $V_k^* = \{v \in V_k : s_A(v) > 1 + \varepsilon\}$.
If $V_k^* = \emptyset$, then stop. Set

$$E_\varepsilon = \bigcup_{v \in V_k} \{x \in X : \langle f(x), v \rangle \geq 1\}$$

(see also comment 3 of Section 5.3). Otherwise, go to the next step.

Step 4. Set

$$S_{k+1} = S_k \cap \{z \in \mathbb{R}^m : \langle z, y_v \rangle \leq 1, \ v \in V_k^*\}$$

and find the vertex set V_{k+1} of S_{k+1}. Set $k = k + 1$ and return to Step 2.

The convergence of this algorithm is seen in the next proposition.

Theorem 5.5 *Assume that $f(X) \subseteq \operatorname{int} \mathbb{R}^m_+$ and $[f(X)]^{\diamond}$ is a nonempty, compact and convex set. Then*

(i) *for a given $\varepsilon > 0$ the algorithm terminates after a finite number of iterations and*

$$S(\mathrm{VP}) \subseteq E_\varepsilon \subseteq S_{\varepsilon\delta_k}(\mathrm{VP}),$$

where $\delta_k = 1/\left(\min_{v \in V_k} \sum_{i=1}^m v_i\right)$;

(ii) *if problem* (VP) *is linear, then the algorithm terminates after a finite number of iterations with zero tolerance $\varepsilon = 0$ and*

$$S(\mathrm{VP}) = \bigcup_{v \in V_k} \{x \in X : \langle f(x), v \rangle = 1\}.$$

Proof It follows from Theorem 5.3 that at some iteration k we must have $V_k^* = \emptyset$ and the algorithm terminates after a finite number of iterations. Let $x \in S(\mathrm{VP})$. If $f(x) \in A \setminus B_k$, then as in the proof of Theorem 5.3, there are some $v \in V_k$ and $y \in B_k$ such that $\langle v, y \rangle = 1$ and $f(x) = \langle v, f(x) \rangle y$. Since $\langle v, f(x) \rangle > 1$, one has $x \in E_\varepsilon$. Now let $x \in E_\varepsilon$. There exists $v \in V_k$ such that $\langle v, f(x) \rangle \geq 1$. It follows from the condition $V_k^* = \emptyset$ that for all $x' \in X$

$$\langle v, f(x') \rangle \leq 1 + \varepsilon \leq \langle v, f(x) \rangle + \varepsilon. \tag{5.2}$$

If x is not an $\varepsilon\delta_k$-solution of (VP), one can find $x_0 \in X$ and $c \in \mathrm{int}\mathbb{R}^m_+$ such that

$$f(x_0) = f(x) + (\varepsilon\delta_k, \ldots, \varepsilon\delta_k) + c.$$

Then one has

$$\langle v, f(x_0)\rangle > \langle v, f(x)\rangle + \varepsilon\delta_k \sum_{i=1}^{m} v_i \geq \langle v, f(x)\rangle + \varepsilon,$$

which contradicts (5.2). When (VP) is linear, at each iteration k (with $\varepsilon = 0$), the polytope B_k contains new vertices of $f(X)$. Since the number of vertices of $f(X)$ is finite, the algorithm terminates after a finite number of iterations. □

We would like to point out that when the algorithm terminates, in view of Theorem 5.5, all elements of the set E_ε are $\varepsilon\delta_k$-solutions of (VP). Moreover, since A contains the origin of the space in its interior, there is a positive γ such that all coordinates of elements of V_k are greater then γ. Consequently, δ_k is majorized by $1/(m\gamma)$ and $\varepsilon\delta_k$ converges to 0 as soon as ε tends to 0.

5.2 Illustrative examples

We illustrate our method by two small examples. Consider the following biobjective programming problem:

$$\max(-(0.5(x_1 - 1)^2 + (x_2 - 5)^2), -(x_1 - x_2)^2)$$

s.t. $x_1 \geq 0$, $x_2 \geq 0$ and $x_1^2 + x_2^2 \leq 4$.

First, we find $a_1 = -9.3756$, $a_2 = 0$, $b_1 = -26.0$ and $b_2 = -4.0$. At the initialization the polyhedron S_1 is defined by

$$S_1 = \{\xi \in \mathbb{R}^2_+ : \langle \xi, y_1\rangle \leq 1, \langle \xi, y_2\rangle \leq 1\},$$

where

$$y_1 = (0, 4.0000), \quad y_2 = (16.6244, 0).$$

The vertex set V_1 consist of three elements:

$$v_0 = (0, 0), \quad v_1 = (0.0602, 0), \quad v_2 = (0, 0.2500), \quad v_3 = (0.0602, 0.2500).$$

By solving (P_v) for each v_1, v_2 and v_3, we obtain

$$s_A(v_1) = 1, \quad s_A(v_2) = 1$$

and

$$s_A(v_3) = 1.8277, \quad y_3 = (14.3313, 3.8624).$$

It is obvious that $V_1^* = \{v_3\}$. The cut that uses y_3 generates the polyhedron S_2 with two new vertices

$$v_4 = (0.0024, 0.2500), \quad v_5 = (0.0602, 0.0357).$$

By solving (P_v) for v_4 and v_5 we obtain

$$s_A(v_4) = 1.0670, \quad y_4 = (16.2179, 2.5611)$$

and

$$s_A(v_5) = 1.0314, \quad y_5 = (13.1245, 3.9997).$$

Two new cuts using y_4 and y_5 will generate the polyhedron S_3 with three new vertices

$$v_6 = (0.0602, 0.0095), \quad v_7 = (0.0502, 0.0727), \quad v_8 = (0.0207, 0.1821).$$

The first three iterations are illustrated in the Fig 1. With $\varepsilon = 0.001$ the algorithm terminates after six iterations.

Figure 3 shows the set E_ε (Step 3) in the decision space which is an approximation of the weak efficient solutions of the problem. This set consists of the intersections of the regions bounded by elliptic upper level sets $\langle v, f(x)\rangle \geq 1$ and the feasible region (the first quarter disc)

$$\{(x_1, x_2) \in \mathbb{R}^2 \colon 0 \leq x_1 \leq x_2, \quad x_1^2 + x_2^2 = 4\}.$$

Figure 2 illustrates the cuts performed in the dual space and the solutions of (P_v) which present an approximation of the weak efficient values of the problem.

The second example is the following problem with three criteria

$$\max(-(0.5(x_1 - 1)^2 + (x_2 - 5)^2), -(x_1 - x_2)^2, -(x_1^4 + x_2^4))$$

s.t. $x_1 \geq 0$, $x_2 \geq 0$ and $x_1^2 + x_2^2 \leq 4$.

With $\varepsilon = 0.001$, the algorithm terminates after seven iterations and 33.98 s. The final results are exhibited in Figs 4 and 5 below.

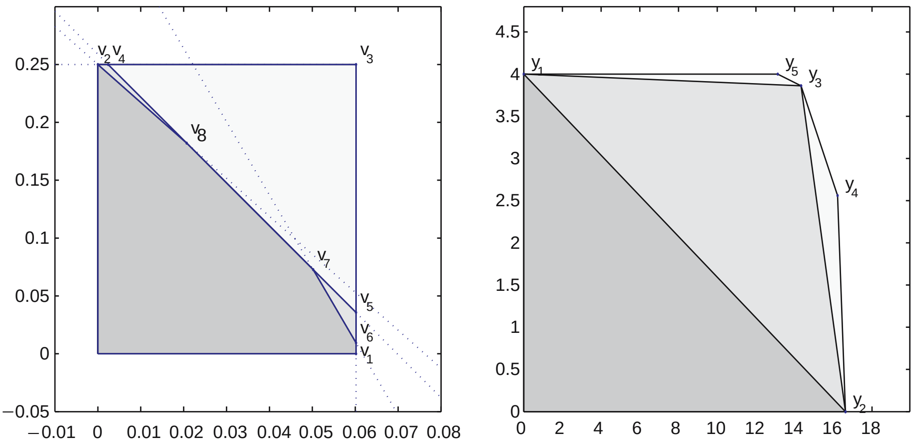


Fig. 1 Left figure presents the cuts and polyhedra S_k described in Step 4, and the right figure presents the points obtained by solving (P_v) (Step 2) and polytopes B_k

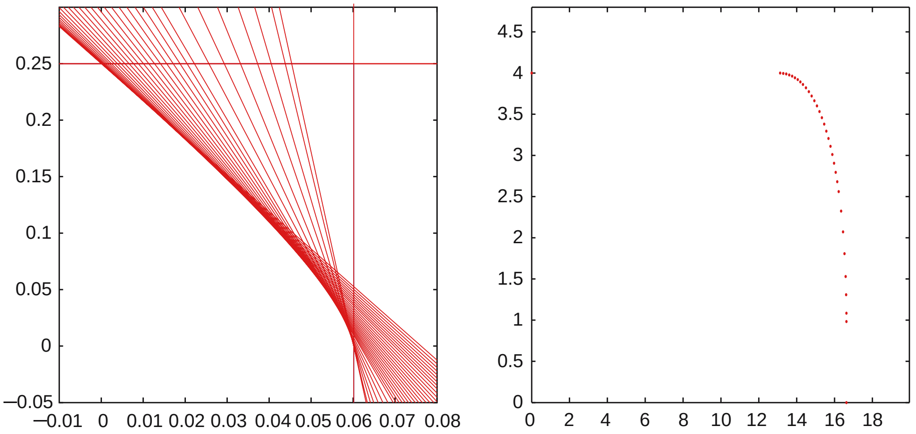


Fig. 2 On the left the straight lines present the cuts described in Step 4, and on the right the points y_v obtained in Step 2 give an approximation of the weak efficient values of the problem in the outcome space

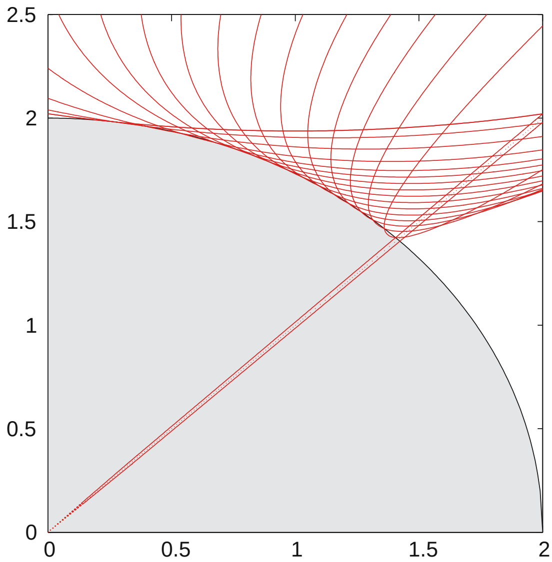


Fig. 3 The set E_ε obtained in Step 3 is an approximation of the weak efficient solution set in the decision space

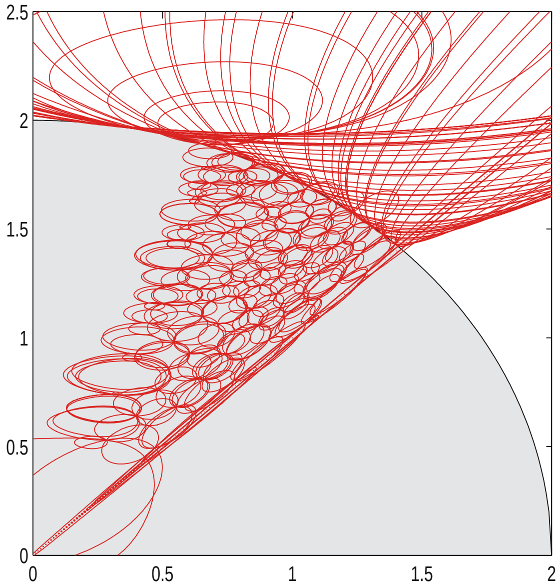
5.3 Comments

We close up this section by some comments on our algorithm.

1. The problem (P_v) in Step 2 can explicitly be written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} v_i y_i, \\ \text{s. t.} \quad & y \geq 0, \\ & f(x) - y \geq 0, \ x \in X. \end{aligned}$$

Fig. 4 The set $E_{\mathcal{E}}$ is given by the intersections of the regions bounded by elliptic upper level sets $\langle v, f(x)\rangle \geq 1$ and the feasible region



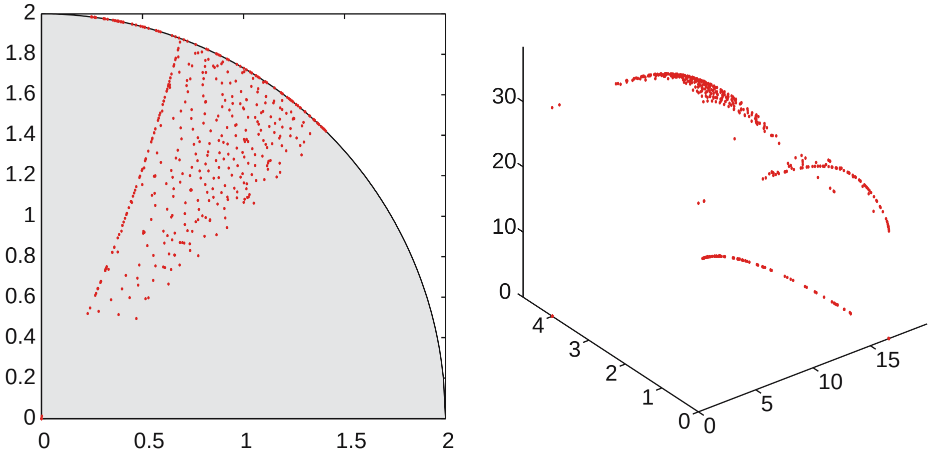


Fig. 5 The left figure presents the set $\hat{E}_{\mathcal{E}}$ (see Comment 3) which is the part of the weak efficient solution set S(VP) and nicely distributed among $E_{\mathcal{E}}$. The right figure shows the set of the points $y_v, v \in V_k$ in the outcome space

This is a problem of minimizing a linear function on a convex set when X is a convex set and f is a concave function. In our examples, we solve (P_v) by using the optimization toolbox of MATLAB version 7.

2. In Step 4, the vertex set V_{k+1} can be derived from V_k by the method of [4]. Other methods for numerating the vertices of a polyhedron can be used as well.
3. In Step 3, the computation of the set $E_{\mathcal{E}}$ is very costly when X and f have no particular (simple) structure. In general, one finds some $x_v \in X$ such that $y_v = f(x_v)$ solves Problem (P_v) and stores the set $\hat{E}_{\mathcal{E}} = \{x_v : v \in V_k\}$. This set is the best representative part of the solution set S(VP) in the following sense:

(i) $\hat{E}_\varepsilon \subseteq S(\text{VP})$.
(ii) For each $x \in S(\text{VP})$, there is some $x_v \in \hat{E}_\varepsilon$ such that

$$|\langle v, f(x_v)\rangle - \langle v, f(x)\rangle| \leq \varepsilon,$$

where v is a vertex of the polyhedron $\hat{S}_k$ that approximates the polar of $[f(X)]^\diamond$ from outside. Indeed, (i) is obtained from the fact that the vector v is positive, hence the function $\langle v, .\rangle$ is increasing on $\mathbb{R}^m$. For (ii), let $x \in S(\text{VP})$. By Theorem 5.5, there is $v \in V_k$ such that $\langle v, f(x)\rangle \geq 1$. When the algorithm terminates, one has $V_k^* = \emptyset$, so that $\langle v, y_v)\rangle \leq 1 + \varepsilon$ for all $v \in V_k$. Let $y_v \in [f(X)]^\diamond$ be a solution of (P_v) and let $x_v \in X$ be such that $f(x_v) = y_v$. Then

$$1 + \varepsilon \geq \langle v, f(x_v)\rangle \geq \langle v, f(x)\rangle \geq 1,$$

which implies (ii).

4. The small value of ε to choose in the initialization step is the precision of the solutions we wish to obtain by the algorithm. It allows us to stop the algorithm after a finite number of iterations. In general the number of iterations needed to generate the efficient set depends heavily on the structure of the problem and on the value of ε, so that it is very difficult to determine an upper bound for this number when ε is given.
5. In a previous work [14] we have developed an algorithm, referred to as ALG 1 to solve (VP) by building up a sequence of polytopes A_k to approximate A from outside. In that algorithm the polytope A_{k+1} is obtained from A_k by cutting planes $\langle v, y\rangle \leq \rho_v$, where v is the unit normal vector to A from a vertex of A_k. The algorithm of the present paper, referred to as ALG 2, attempts to approximate from outside the polar set A° by a sequence of polyhedra $\hat{S}_k$. By polarity, the sequence of polars $(\hat{S}_k)^\circ$ approximates the set A from the interior. We notice, however, that ALG 1 is performed in the space of outcomes $\mathbb{R}^m$, while ALG 2 is carried out partly in the dual space of $\mathbb{R}^m$. For this reason, ALG 2 can be regarded as a *primal-dual* algorithm.
6. In [14] we have already discussed the advantages of ALG 1 over some existing algorithms. It is worthwhile noticing that both ALGs 1 and 2 allow to obtain a set of ε-solutions which contains the set of exact solutions after a finite number of iterations with a given $\varepsilon > 0$. This property is not guaranteed by most existing algorithms. For instance by using the normal-boundary intersection method of [5], it is impossible to choose a finite set of CHIM simplex in order to obtain an approximate solution set within a small tolerance (see Section 5.4 of [14]).
7. From the computational point of view, both algorithms ALGs 1 and 2 have a common feature at each iteration: solve optimization problems over the set $[f(X)]^\diamond$. However, the objective functions of ALG 1 are quadratic functions, while the objective functions of ALG 2 are linear. This shows that ALG 2 requires less computation time than ALG 1 does. A detailed comparison of numerical results of these two algorithms and some others is beyond of the scope of this paper and will be addressed in a forthcoming report.

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